CHAPTER 10
Limit Formulas

10.1 Definition of Limit

LIMIT OF A FUNCTION (INFORMAL DEFINITION)

The notation

$$\lim_{x \to c} f(x) = L$$

is read ''the limit of $f(x)$ as x approaches c is L'' and means that the functional values $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to c.

LIMIT OF A FUNCTION (FORMAL DEFINITION)

The limit statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\epsilon > 0$, there corresponds a number $\delta > 0$ with the property that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - c| < \delta$$

A FUNCTION DIVERGES TO INFINITY (INFORMAL DEFINITION)

A function f that increases or decreases without bound as x approaches c is said to **diverge to infinity** (∞) at c. We indicate this behavior by writing

(*continued*)

$$\lim_{x \to c} f(x) = +\infty$$

if x increases without bound and by

$$\lim_{x \to c} f(x) = -\infty$$

if x decreases without bound.

INFINITE LIMIT (FORMAL DEFINITION)

We write $\lim_{x \to c} f(x) = +\infty$ if, for any number $N > 0$ (no matter how large), it is possible to find a number $\delta > 0$ such that $f(x) > N$ whenever $0 < |x - c| < \delta$.

LIMITS INVOLVING INFINITY

The limit statement $\lim_{x \to +\infty} f(x) = L$ means that for any number $\epsilon > 0$, there exists a number N_1 such that

$$|f(x) - L| < \epsilon \text{ whenever } x > N_1$$

for x in the domain of f. Similarly $\lim_{x \to -\infty} f(x) = M$ means that for any $\epsilon > 0$, there exists a number N_2 such that

$$|f(x) - M| < \epsilon \text{ whenever } x < N_2$$

LIMIT OF A FUNCTION OF TWO VARIABLES (INFORMAL DEFINITION)

The notation

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

(*continued*)

means that the functional values $f(x, y)$ can be made arbitrarily close to L by choosing the point (x, y) close to the point (x_0, y_0).

LIMIT OF A FUNCTION OF TWO VARIABLES (FORMAL DEFINITION)

Suppose the point $P_0(x_0, y_0)$ has the property that every disk centered at P_0 contains at least one point in the domain of f other than P_0 itself. Then the number L is the **limit of f at P** if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon \text{ whenever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

In this case, we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

10.2 Rules of Limits

BASIC RULES

For any real numbers a and c, suppose the functions f and g both have limits at $x = c$. Suppose also that both $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist.

Limit of a constant	$\lim_{x\to c} k = k$ for any constant k
Limit of x	$\lim_{x\to c} x = c$
Scalar rule	$\lim_{x\to c}[af(x)] = a\lim_{x\to c} f(x)$
Sum rule	$\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
Difference rule	$\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
Linearity rule	$\lim_{x\to+\infty}[af(x) + bg(x)] = a\lim_{x\to+\infty} f(x) + b\lim_{x\to+\infty} g(x)$

Product rules $\lim_{x\to c}[f(x)g(x)] = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)]$
$\lim_{x\to+\infty}[f(x)g(x)] =$
$[\lim_{x\to+\infty} f(x)]\,[\lim_{x\to+\infty} g(x)]$

Quotient rules $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$
$\lim_{x\to+\infty} \dfrac{f(x)}{g(x)} =$
$\dfrac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)}$ if $\lim_{x\to+\infty} g(x) \neq 0$

Power rules $\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$ n is a rational number
$\lim_{x\to+\infty}[f(x)]^n = [\lim_{x\to+\infty} f(x)]^n$

Limit limitation theorem Suppose $\lim_{x\to c} f(x)$ exists and $f(x) \geq 0$ throughout an open interval containing the number c, except possibly at c itself. Then $\lim_{x\to c} f(x) \geq 0$.

The squeeze rule If $g(x) \leq f(x) \leq h(x)$ for all x in an open interval containing c (except possibly at c itself) and if

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then $\lim_{x\to c} f(x) = L$.

Limits to infinity $\lim_{x\to+\infty} \dfrac{A}{x^n} = 0$ and $\lim_{x\to-\infty} \dfrac{A}{x^n} = 0$

Infinite-limit theorem If $\lim_{x\to c} f(x) = +\infty$ and $\lim_{x\to c} g(x) = A$, then

$\lim_{x\to c}[f(x)g(x)] = +\infty$ and $\lim_{x\to c} \dfrac{f(x)}{g(x)} = +\infty$ if $A > 0$

$\lim_{x\to c}[f(x)g(x)] = -\infty$ and $\lim_{x\to c} \dfrac{f(x)}{g(x)} = -\infty$ if $A < 0$

l'Hôpital's rule Let f and g be differentiable functions on an open interval containing c (except possibly at c itself).
If $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ produces an indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right side exists.

TRIGONOMETRIC LIMITS

$$\lim_{x \to c} \cos x = \cos c \quad \lim_{x \to c} \sec x = \sec c$$

$$\lim_{x \to c} \sin x = \sin c \quad \lim_{x \to c} \csc x = \csc c$$

$$\lim_{x \to c} \tan x = \tan c \quad \lim_{x \to c} \cot x = \cot c$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \; \lim_{x \to 0} \frac{\sin ax}{x} = a \; \lim_{x \to 0} \frac{\tan x}{x} = 1 \; \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

MISCELLANEOUS LIMITS

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e \qquad \lim_{n \to 0} (1 + n)^{1/n} = e$$

$$\lim_{n \to +\infty} \left(1 + \frac{k}{n}\right)^n = e^k \quad \lim_{n \to +\infty} p \left(1 + \frac{1}{n}\right)^{nt} = pe^t$$

$$\lim_{n \to +\infty} n^{1/n} = 1$$

10.3 Limits of a Function of Two Variables

BASIC FORMULAS AND RULES FOR LIMITS OF A FUNCTION OF TWO VARIABLES

Suppose $\lim_{(x,y) \to (x_0,y_0)} f(x, y)$ and $\lim_{(x,y) \to (x_0,y_0)} g(x, y)$ both exist, with $\lim_{(x,y) \to (x_0,y_0)} f(x, y) = L$ and $\lim_{(x,y) \to (x_0,y_0)} g(x, y) = M$.
Then the following rules obtain:

Scalar rule

$$\lim_{(x,y) \to (x_0,y_0)} [af(x, y)] = a \lim_{(x,y) \to (x_0,y_0)} f(x, y) = aL$$

Sum rule

$$\lim_{(x,y) \to (x_0,y_0)} [f + g](x, y) = \left[\lim_{(x,y) \to (x_0,y_0)} f(x, y)\right] + \left[\lim_{(x,y) \to (x_0,y_0)} g(x, y)\right] = L + M$$

Product rule $\lim_{(x,y)\to(x_0,y_0)} [fg](x, y)$

$$= \left[\lim_{(x,y)\to(x_0,y_0)} f(x, y)\right]\left[\lim_{(x,y)\to(x_0,y_0)} g(x, y)\right]$$

$$= LM$$

Quotient rule $$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f}{g}\right](x, y) = \frac{\lim_{(x,y)\to(x_0,y_0)} f(x, y)}{\lim_{(x,y)\to(x_0,y_0)} g(x, y)} = \frac{L}{M}$$

if $M \neq 0$

Substitution rule

If $f(x, y)$ is a polynomial or a rational function, limits may be found by substituting for x and y (excluding values that cause division by zero).