ALGEBRA

Lines

Slope of the line through $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Point-slope equation of line through $P_1 = (x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Point-point equation of line through $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$y - y_1 = m(x - x_1) \quad \text{where } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Lines of slope m_1 and m_2 are parallel if and only if $m_1 = m_2$.
Lines of slope m_1 and m_2 are perpendicular if and only if $m_1 = -\frac{1}{m_2}$.

Circles

Equation of the circle with center (a, b) and radius r:

$$(x - a)^2 + (y - b)^2 = r^2$$

Distance and Midpoint Formulas

Distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\overline{P_1 P_2}$: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

Laws of Exponents

$x^m x^n = x^{m+n}$ $\qquad \frac{x^m}{x^n} = x^{m-n}$ $\qquad (x^m)^n = x^{mn}$

$x^{-n} = \frac{1}{x^n}$ $\qquad (xy)^n = x^n y^n$ $\qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

$x^{1/n} = \sqrt[n]{x}$ $\qquad \sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$ $\qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$

$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

Special Factorizations

$x^2 - y^2 = (x + y)(x - y)$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Binomial Theorem

$(x + y)^2 = x^2 + 2xy + y^2$

$(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + nxy^{n-1} + y^n$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot k}$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

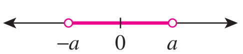
If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

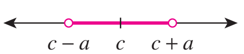
If $a < b$ and $c < 0$, then $ca > cb$.

$|x| = x \quad$ if $x \geq 0$

$|x| = -x \quad$ if $x \leq 0$



$|x| < a$ means
$-a < x < a$.



$|x - c| < a$ means
$c - a < x < c + a$.

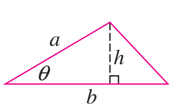
GEOMETRY

Formulas for area A, circumference C, and volume V

Triangle

$A = \frac{1}{2}bh$

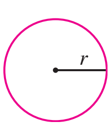
$\quad = \frac{1}{2}ab \sin \theta$



Circle

$A = \pi r^2$

$C = 2\pi r$

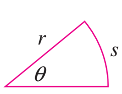


Sector of Circle

$A = \frac{1}{2}r^2\theta$

$s = r\theta$

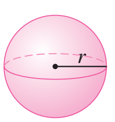
(θ in radians)



Sphere

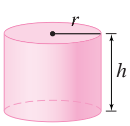
$V = \frac{4}{3}\pi r^3$

$A = 4\pi r^2$



Cylinder

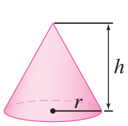
$V = \pi r^2 h$



Cone

$V = \frac{1}{3}\pi r^2 h$

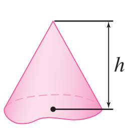
$A = \pi r\sqrt{r^2 + h^2}$



Cone with arbitrary base

$V = \frac{1}{3}Ah$

where A is the area of the base



Pythagorean Theorem: For a right triangle with hypotenuse of length c and legs of lengths a and b, $c^2 = a^2 + b^2$.

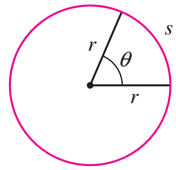
TRIGONOMETRY

Angle Measurement

π radians $= 180°$

$1° = \dfrac{\pi}{180}$ rad $\qquad 1$ rad $= \dfrac{180°}{\pi}$

$s = r\theta \quad$ (θ in radians)

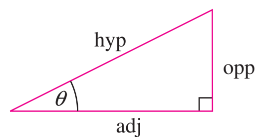


Right Triangle Definitions

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \dfrac{\text{adj}}{\text{hyp}}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\text{opp}}{\text{adj}} \qquad \cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\text{adj}}{\text{opp}}$

$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\text{hyp}}{\text{adj}} \qquad \csc\theta = \dfrac{1}{\sin\theta} = \dfrac{\text{hyp}}{\text{opp}}$

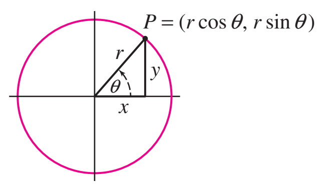


Trigonometric Functions

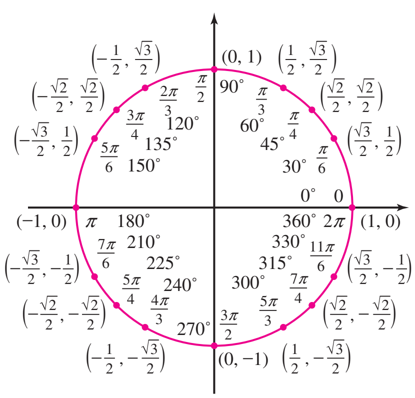
$\sin\theta = \dfrac{y}{r} \qquad \csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r} \qquad \sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$



$\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1 \qquad \lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta} = 0$



Fundamental Identities

$\sin^2\theta + \cos^2\theta = 1 \qquad \sin(-\theta) = -\sin\theta$

$1 + \tan^2\theta = \sec^2\theta \qquad \cos(-\theta) = \cos\theta$

$1 + \cot^2\theta = \csc^2\theta \qquad \tan(-\theta) = -\tan\theta$

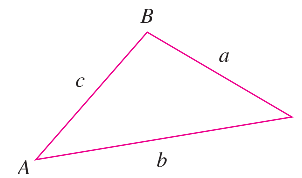
$\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta \qquad \sin(\theta + 2\pi) = \sin\theta$

$\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$

$\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta \qquad \tan(\theta + \pi) = \tan\theta$

The Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Addition and Subtraction Formulas

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

Double-Angle Formulas

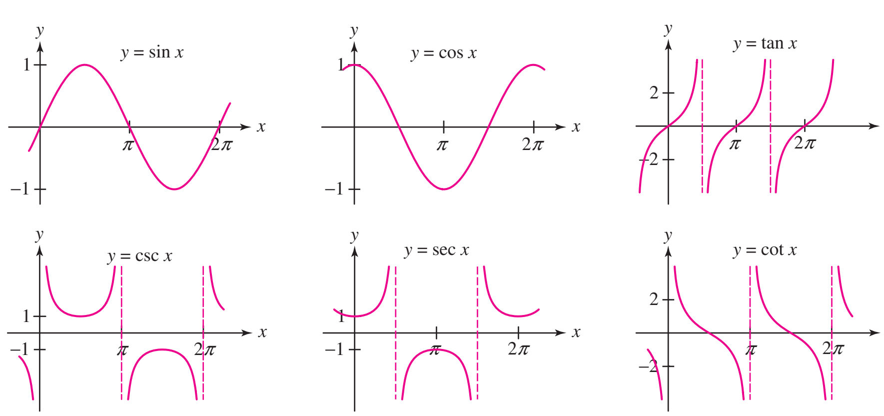
$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$\sin^2 x = \dfrac{1 - \cos 2x}{2} \qquad \cos^2 x = \dfrac{1 + \cos 2x}{2}$

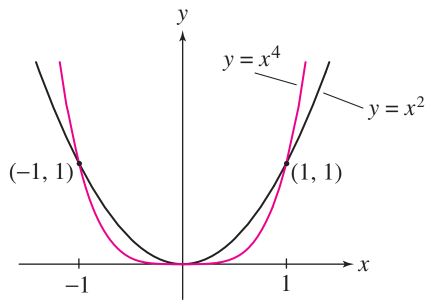
Graphs of Trigonometric Functions



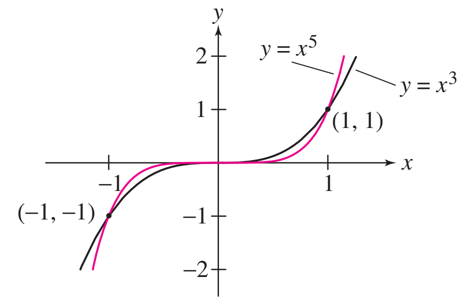
ELEMENTARY FUNCTIONS

Power Functions $f(x) = x^a$

$f(x) = x^n$, n a positive integer

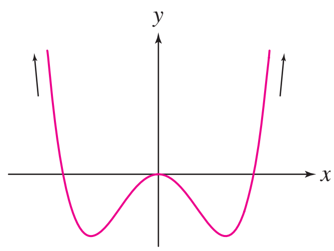


n even



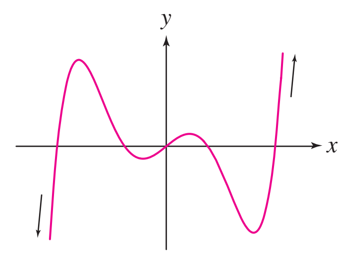
n odd

Asymptotic behavior of a polynomial function of even degree and positive leading coefficient



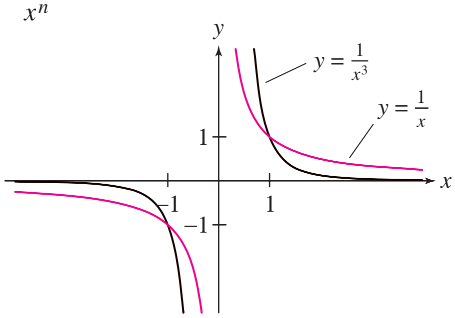
n even

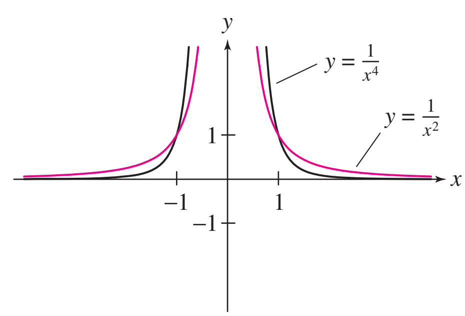
Asymptotic behavior of a polynomial function of odd degree and positive leading coefficient



n odd

$f(x) = x^{-n} = \dfrac{1}{x^n}$

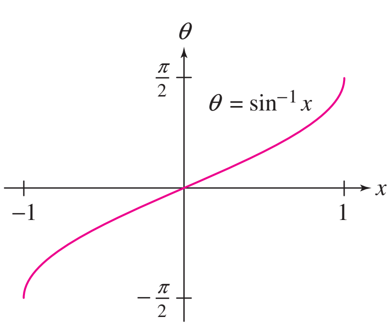




Inverse Trigonometric Functions

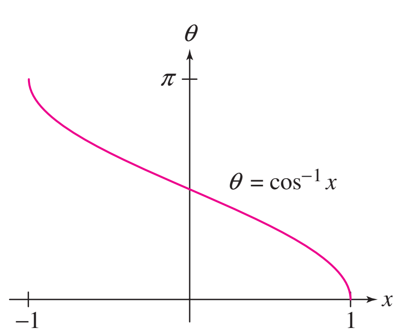
$\arcsin x = \sin^{-1} x = \theta$

$\Leftrightarrow \quad \sin\theta = x, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$



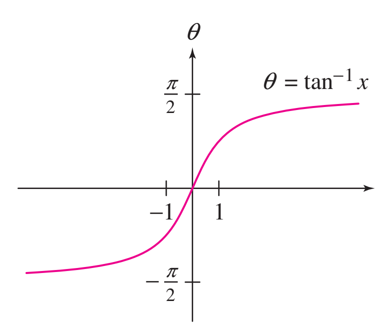
$\arccos x = \cos^{-1} x = \theta$

$\Leftrightarrow \quad \cos\theta = x, \quad 0 \le \theta \le \pi$



$\arctan x = \tan^{-1} x = \theta$

$\Leftrightarrow \quad \tan\theta = x, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$



Exponential and Logarithmic Functions

$\log_a x = y \quad \Leftrightarrow \quad a^y = x$

$\ln x = y \quad \Leftrightarrow \quad e^y = x$

$\log_a(xy) = \log_a x + \log_a y$

$\log_a(a^x) = x \qquad a^{\log_a x} = x$

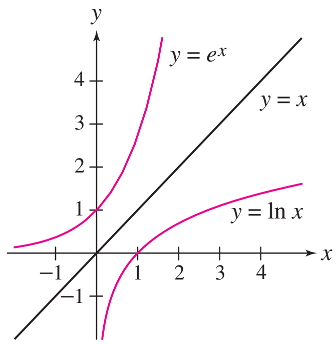
$\ln(e^x) = x \qquad e^{\ln x} = x$

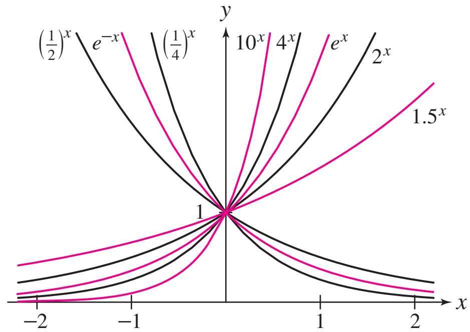
$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

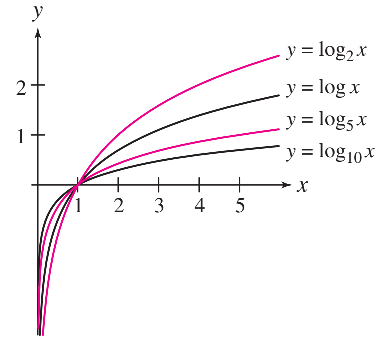
$\log_a 1 = 0 \qquad \log_a a = 1$

$\ln 1 = 0 \qquad \ln e = 1$

$\log_a(x^r) = r \log_a x$







$\lim\limits_{x\to\infty} a^x = \infty, \quad a > 1$

$\lim\limits_{x\to\infty} a^x = 0, \quad 0 < a < 1$

$\lim\limits_{x\to-\infty} a^x = 0, \quad a > 1$

$\lim\limits_{x\to-\infty} a^x = \infty, \quad 0 < a < 1$

$\lim\limits_{x\to 0^+} \log_a x = -\infty$

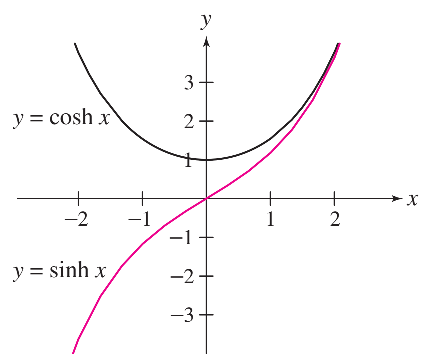
$\lim\limits_{x\to\infty} \log_a x = \infty$

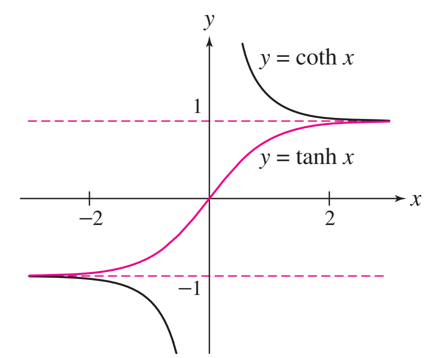
Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \dfrac{1}{\sinh x}$

$\cosh x = \dfrac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \dfrac{1}{\cosh x}$

$\tanh x = \dfrac{\sinh x}{\cosh x} \qquad \coth x = \dfrac{\cosh x}{\sinh x}$





$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

Inverse Hyperbolic Functions

$y = \sinh^{-1} x \quad \Leftrightarrow \quad \sinh y = x$

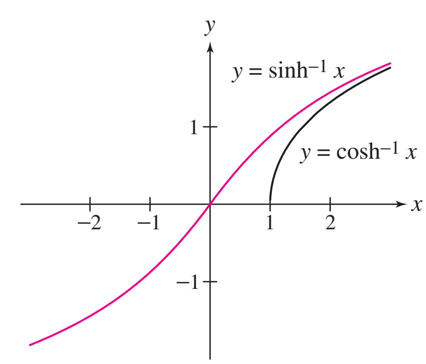
$y = \cosh^{-1} x \quad \Leftrightarrow \quad \cosh y = x \text{ and } y \geq 0$

$y = \tanh^{-1} x \quad \Leftrightarrow \quad \tanh y = x$

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x > 1$

$\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1 + x}{1 - x}\right) \qquad -1 < x < 1$



DIFFERENTIATION

Differentiation Rules

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}x = 1$

3. $\frac{d}{dx}(x^n) = nx^{n-1}$ (Power Rule)

4. $\frac{d}{dx}[cf(x)] = cf'(x)$

5. $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

6. $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ (Product Rule)

7. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ (Quotient Rule)

8. $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (Chain Rule)

9. $\frac{d}{dx}f(x)^n = nf(x)^{n-1}f'(x)$ (General Power Rule)

10. $\frac{d}{dx}f(kx+b) = kf'(kx+b)$

11. $g'(x) = \frac{1}{f'(g(x))}$ where $g(x)$ is the inverse $f^{-1}(x)$

12. $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

Trigonometric Functions

13. $\frac{d}{dx}\sin x = \cos x$

14. $\frac{d}{dx}\cos x = -\sin x$

15. $\frac{d}{dx}\tan x = \sec^2 x$

16. $\frac{d}{dx}\csc x = -\csc x \cot x$

17. $\frac{d}{dx}\sec x = \sec x \tan x$

18. $\frac{d}{dx}\cot x = -\csc^2 x$

Inverse Trigonometric Functions

19. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

20. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

21. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

22. $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

23. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

24. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

Exponential and Logarithmic Functions

25. $\frac{d}{dx}(e^x) = e^x$

26. $\frac{d}{dx}(a^x) = (\ln a)a^x$

27. $\frac{d}{dx}\ln|x| = \frac{1}{x}$

28. $\frac{d}{dx}(\log_a x) = \frac{1}{(\ln a)x}$

Hyperbolic Functions

29. $\frac{d}{dx}(\sinh x) = \cosh x$

30. $\frac{d}{dx}(\cosh x) = \sinh x$

31. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

32. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

33. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

34. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Inverse Hyperbolic Functions

35. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

36. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$

37. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

38. $\frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$

39. $\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$

40. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

INTEGRATION

Substitution

If an integrand has the form $f(u(x))u'(x)$, then rewrite the entire integral in terms of u and its differential $du = u'(x)\,dx$:

$$\int f(u(x))u'(x)\,dx = \int f(u)\,du$$

Integration by Parts Formula

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

TABLE OF INTEGRALS

Basic Forms

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

Exponential and Logarithmic Forms

17. $\int ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$
18. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$
19. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a\sin bu - b\cos bu) + C$
20. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a\cos bu + b\sin bu) + C$
21. $\int \ln u\,du = u\ln u - u + C$
22. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$
23. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Forms

24. $\int \sinh u\,du = \cosh u + C$
25. $\int \cosh u\,du = \sinh u + C$
26. $\int \tanh u\,du = \ln\cosh u + C$
27. $\int \coth u\,du = \ln|\sinh u| + C$
28. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$
29. $\int \operatorname{csch} u\,du = \ln\left|\tanh\frac{1}{2}u\right| + C$
30. $\int \operatorname{sech}^2 u\,du = \tanh u + C$
31. $\int \operatorname{csch}^2 u\,du = -\coth u + C$
32. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$
33. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Trigonometric Forms

34. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
35. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
36. $\int \tan^2 u\,du = \tan u - u + C$
37. $\int \cot^2 u\,du = -\cot u - u + C$
38. $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$
39. $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$
40. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$
41. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$
42. $\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

43. $\int \csc^3 u\,du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

44. $\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

45. $\int \cos^n u\,du = \frac{1}{2}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

46. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

47. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

48. $\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

49. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

50. $\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

51. $\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

52. $\int \sin au \cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

53. $\int u \sin u\,du = \sin u - u\cos u + C$

54. $\int u \cos u\,du = \cos u + u \sin u + C$

55. $\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$

56. $\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$

57. $\int \sin^n u \cos^m u\,du$

$$= -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u\,du$$

$$= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u\,du$$

Inverse Trigonometric Forms

58. $\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

59. $\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

60. $\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

61. $\int u\sin^{-1} u\,du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

62. $\int u\cos^{-1} u\,du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u\tan^{-1} u\,du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

64. $\int u^n \sin^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

65. $\int u^n \cos^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n \tan^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

Forms Involving $\sqrt{a^2 - u^2}$, $a > 0$

67. $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

68. $\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

69. $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

70. $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$

71. $\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

72. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

73. $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$

74. $\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

75. $\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

Forms Involving $\sqrt{u^2 - a^2}$, $a > 0$

76. $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

77. $\int u^2\sqrt{u^2-a^2}\,du$

$$= \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

78. $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$

79. $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

80. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

81. $\int \frac{u^2\,du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

82. $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

83. $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

Forms Involving $\sqrt{a^2 + u^2}$, $a > 0$

84. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

85. $\int u^2\sqrt{a^2+u^2}\,du$

$$= \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$$

86. $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

87. $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

88. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

89. $\displaystyle\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

90. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

91. $\displaystyle\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

92. $\displaystyle\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Forms Involving $a + bu$

93. $\displaystyle\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a+bu - a\ln|a+bu|\right) + C$

94. $\displaystyle\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

95. $\displaystyle\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

96. $\displaystyle\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

97. $\displaystyle\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

98. $\displaystyle\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

99. $\displaystyle\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

100. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

101. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

102. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

103. $\displaystyle\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

104. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$

$\displaystyle = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$

105. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

106. $\displaystyle\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

107. $\displaystyle\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

Forms Involving $\sqrt{2au - u^2}$, $a > 0$

108. $\displaystyle\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

109. $\displaystyle\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

110. $\displaystyle\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$

111. $\displaystyle\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$

ESSENTIAL THEOREMS

Intermediate Value Theorem

If $f(x)$ is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value $c \in (a, b)$ such that $f(c) = M$.

Mean Value Theorem

If $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there exists at least one value $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Extreme Values on a Closed Interval

If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains both a minimum and a maximum value on $[a, b]$. Furthermore, if $c \in [a, b]$ and $f(c)$ is an extreme value (min or max), then c is either a critical point or one of the endpoints a or b.

The Fundamental Theorem of Calculus, Part I

Assume that $f(x)$ is continuous on $[a, b]$ and let $F(x)$ be an antiderivative of $f(x)$ on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Fundamental Theorem of Calculus, Part II

Assume that $f(x)$ is a continuous function on $[a, b]$. Then the area function $A(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$, that is,

$$A'(x) = f(x) \quad \text{or equivalently} \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Furthermore, $A(x)$ satisfies the initial condition $A(a) = 0$.